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Numerical Method for Tokamak Equilibrium with Outside Limiter

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A numerical scheme is developed which solves the axisymmetric tokamak equilibrium under the constraints of prescribed values of external coil currents and plasma parameters for a plasma with its surface in contact with an outside limiter. The application of this method to the tokamak equilibria with high poloidal beta and/or non-circular cross section or with divertor configuration of which stagnation point is located outside of the torus is described and discussed. © 1986 Academic Press, Inc.

I. INTRODUCTION

An axisymmetric MHD equilibrium code is a basic numerical tool for the magnetics design, the MHD stability analysis, the experimental data analysis and other problems in a tokamak. Several types of equilibrium codes have been developed to answer these problems. In most of these codes, the vacuum magnetic field is adjusted during iterations so that the plasma position is set to the prescribed one, or the poloidal flux on a certain surface surrounding the plasma column is fixed.

In the problem such as the experimental data analysis, however, we need to solve the equilibrium for prescribed values of external conductor currents, plasma current, poloidal beta value and given positions of material limiters of several types. In this type of the equilibrium problem, it was found that the standard iteration scheme fails to search the equilibrium when plasma column touches a limiter outside of the torus [1, 2]; that is, plasma column continues to shift outward or inward during iterations, depending on the initial guess of the plasma position larger or smaller than the equilibrium one, respectively. The same numerical difficulty occurs when we study the equilibrium of a divertor configuration with the stagnation point (X point) outside of the torus, as in JT-60 tokamak device [3], because the location of the X point is almost constant during iteration for given plasma current and divertor coil current.

Because the numerical phenomena observed above resembles the behaviour of positionally unstable tokamak plasma, the feedback control for positional instability offers the suggestion of a numerical scheme to overcome the above-men-

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tioned problem. Lackner and Hagenow proposed an iteration scheme of enforcing convergence, where they introduced a predominantly vertical field in addition to a given external field and adjusted this field such that the magnetic axis is fixed to a prescribed position during each iteration [4, 5]. This iteration scheme was also effective for regulating the vertical position of the magnetic axis in the configuration with an up–down asymmetry, if a predominantly horizontal field was added to the given external field [4–6]. However, the position of the magnetic axis in equilibrium itself is the unknown factor that we have to determine.

For the same problem, Blum, Le Foll, and Thooris also presented another method in which the Grad–Shafranov equation is solved by Newton iterations with the boundary condition such that the poloidal flux is null on the edge of a computational domain due to the presence of an iron-core transformer [7]. This method is not applicable, however, to the tokamak equilibrium with an air-core transformer, since the poloidal flux on this edge is not always kept constant during the numerical iterations.

In this paper, we propose a new numerical method through which the equilibrium configuration can be obtained for a given external field and prescribed plasma parameters in the tokamak configuration with plasma surface bounded at the material limiter or the X point outside of the torus. The basic idea and the numerical scheme are presented in Section II. Typical results and related discussions are described in Section III.

II. Numerical Scheme

In this section, we describe in detail the numerical scheme to obtain the free-boundary tokamak equilibrium with the following constraints:

(a) Plasma parameters such as plasma current I_p, poloidal beta β_p, and internal inductance l_i have the prescribed values.

(b) Plasma surface contacts with a limiter located outside of the torus on the mid-plane as shown in Fig. 1.

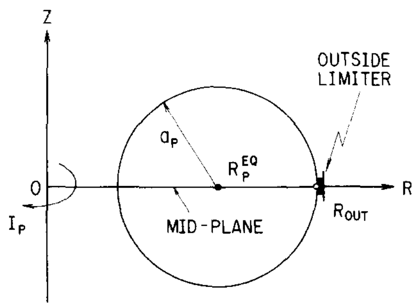

FIG. 1. Tokamak configuration where plasma surface is bounded at an outside limiter located at R_{out} on the mid-plane. It is assumed that plasma current flows clockwise.

(c) The vacuum field to maintain the plasma column in equilibrium is induced only by the given currents of external conductors.

First, we consider the low β_p tokamak with circular cross section in order to make the problem clear. In this case, an equilibrium vacuum field B_z^{vac} is given by the Shafranov's formulation [8]

$$B_z^{\text{vac}} = \frac{\mu_0 I_p}{4\pi R_p}\left(\ln\frac{8R_p}{a_p} - \frac{3}{2} + \beta_p + \frac{l_i}{2}\right), \tag{1}$$

where R_p and a_p are the major radius and the minor radius of the plasma column, respectively. The constraint (b) imposes the relation $a_p = R_{\text{out}} - R_p$ on this equation. Figure 2 shows the schematic dependence of B_z^{vac} on R_p. The equilibrium vacuum field B_z^{vac} increases monotonically with R_p in contrast with the equilibrium where plasma surface contacts with an inside limiter or rail limiters parallel to the mid-plane. This positive gradient of B_z^{vac} is the origin of the numerical instability [1, 2]; that is, if we fix the external field with the vertical component B_z^{ext} and solve the Grad–Shafranov equation with the initial guess for the plasma position R_p^{ref} ($= R_p^{\text{EQ}} + \Delta R_p$) larger than the equilibrium position R_p^{EQ} corresponding to B_z^{ext}, then the plasma column feels the weaker vertical field and is pushed outward and vice versa. To suppress this instability, we introduce a virtual vacuum field ΔB_z and adjust it such as

$$B_z^{\text{vac}}(R_p^{\text{ref}}, Z=0) = B_z^{\text{ext}}(R_p^{\text{ref}}, Z=0) + \Delta B_z, \tag{2}$$

to maintain the plasma column at the reference position R_p^{ref}. For plasma with low β_p and circular cross section, the above virtual vacuum field ΔB_z can be

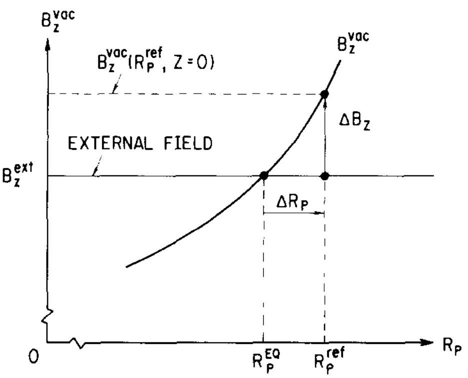

FIG. 2. Equilibrium vacuum field B_z^{vac} as a function of major radius R_p. B_z^{vac} agrees with a given external field B_z^{ext} at the equilibrium position R_p^{EQ}. The virtual vacuum field ΔB_z means a correction field for the displacement ΔR_p of plasma column from R_p^{EQ}.

approximately estimated by linear expansion of the Shafranov's formulation in the vicinity of R_p^{EQ} as follows:

$$\Delta B_z^{\mathrm{analytic}} = \frac{\mu_0 I_p}{4\pi R_p^{\mathrm{EQ}}} \left\{ \frac{R_{\mathrm{out}}}{R_{\mathrm{out}} - R_p^{\mathrm{EQ}}} - \left(\ln \frac{8 R_p^{\mathrm{EQ}}}{R_{\mathrm{out}} - R_p^{\mathrm{EQ}}} - \frac{3}{2} + \beta_p + \frac{l_i}{2} \right) \right\} \frac{\Delta R_p}{R_p^{\mathrm{EQ}}}. \tag{3}$$

It is clear that ΔB_z is a monotonic function of R_p^{ref} or the displacement ΔR_p from R_p^{EQ}, and that ΔB_z must vanish in the equilibrium state. Although the equilibrium position of the plasma column R_p^{EQ} is unknown, the relation between ΔB_z and R_p^{ref} gives us the information whether R_p^{ref} is greater than R_p^{EQ} or not; that is, $R_p^{\mathrm{ref}} > R_p^{\mathrm{EQ}}$ for $\Delta B_z > 0$ and $R_p^{\mathrm{ref}} < R_p^{\mathrm{EQ}}$ for $\Delta B_z < 0$ as illustrated in Fig. 2. Therefore, by checking the sign of ΔB_z and modifying R_p^{ref} appropriately, we can make R_p^{ref} converge to R_p^{EQ}.

Based on the above idea, the numerical procedure presented here consists of two iteration loops; minor iteration loop and major iteration loop. Figure 3 shows the flow chart for the numerical procedure.

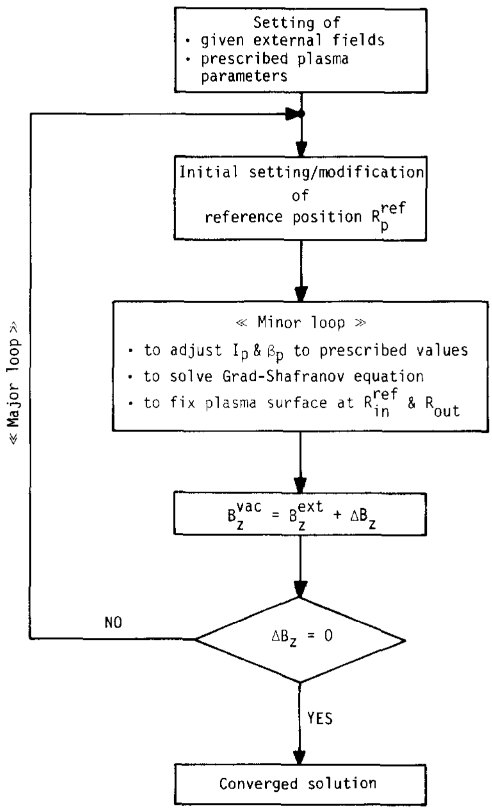

FIG. 3. Flow chart for numerical scheme. Major loop which modifies a reference position R_p^{ref} in accordance with the virtual vacuum field ΔB_z is added to minor iteration loop. R_{in}^{ref} is defined by Eq. (6).

In the minor iteration loop, we solve the Grad–Shafranov equation under the constraints (a) and (b), and regulate the plasma column at the given reference position R_p^{ref} through the iteration scheme of the enforcing convergence [4, 5]. For this regulation of the plasma position, the virtual vacuum field ΔB_z is adjusted so as to satisfy Eq. (2). At first, we solve the Grad–Shafranov equation with only the external field and obtain the poloidal flux $\hat{\psi}^n(R, Z)$, where n is the number of the minor iteration steps. Next, we determine the virtual flux $\Delta\psi^n(R)$ as follows:

$$\Delta\psi^n(R) = -\hat{\psi}^n(R_{\text{in}}^{\text{ref}}, Z=0)\,\frac{R_{\text{out}}^2 - R^2}{R_{\text{out}}^2 - (R_{\text{in}}^{\text{ref}})^2}. \tag{4}$$

The virtual vacuum field with the spatial uniformity ΔB_z^n is given as

$$\Delta B_z^n = \frac{1}{R}\frac{\partial}{\partial R}\Delta\psi^n(R) = \frac{2\hat{\psi}^n(R_{\text{in}}^{\text{ref}}, Z=0)}{R_{\text{out}}^2 - (R_{\text{in}}^{\text{ref}})^2}, \tag{5}$$

where $R_{\text{in}}^{\text{ref}}$ is the innermost location of the reference plasma surface defined by

$$R_{\text{in}}^{\text{ref}} \equiv 2R_p^{\text{ref}} - R_{\text{out}}. \tag{6}$$

Then we add $\Delta\psi^n$ to $\hat{\psi}^n$ and obtain ψ^n:

$$\psi^n(R, Z) = \hat{\psi}^n(R, Z) + \Delta\psi^n(R). \tag{7}$$

The above numerical procedure is schematically illustrated in Fig. 4. It is obvious that the plasma surface passes the two reference points R_{out} and $R_{\text{in}}^{\text{ref}}$ on the mid-

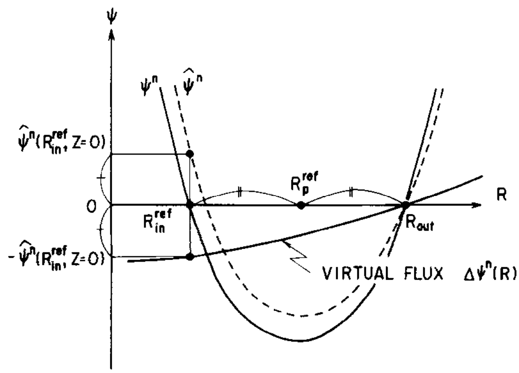

FIG. 4. Schematic illustration of numerical procedure in minor loop. The poloidal flux $\hat{\psi}^n$ is described as

$$\hat{\psi}^n(R, Z) = \psi_p^n(R, Z) + \psi_{\text{ext}}(R, Z) - \psi_p^n(R_{\text{out}}, Z=0) - \psi_{\text{ext}}(R_{\text{out}}, Z=0)$$

where ψ_p^n and ψ_{ext} are the poloidal flux produced by the plasma current and the external conductor currents, respectively. In order to fix the plasma column at the reference position R_p^{ref}, the virtual flux $\Delta\psi^n$ defined by Eq. (4) is added to $\hat{\psi}^n$.

plane; $\psi^n(R_{out}, Z=0)=0$ because of $\hat{\psi}^n(R_{out}, Z=0)=0$ and $\Delta\psi^n(R_{out})=0$, and $\psi^n(R_{in}^{ref}, Z=0)=0$ because of $\Delta\psi^n(R_{in}^{ref})=-\hat{\psi}(R_{in}^{ref}, Z=0)$. This procedure is repeated until ψ^n converges, and we can know the sign of ΔB_z for the given reference position R_p^{ref}.

The virtual vacuum field ΔB_z depends not only on the reference position R_p^{ref} but also on I_p and β_p. The change of I_p and β_p during iterations in the minor loop would disturb the monotonic dependence of ΔB_z on R_p^{ref} as illustrated in Fig. 2, and might break down our convergence scheme. So in the minor loop, I_p and β_p are adjusted to the prescribed values I_p^{set} and β_p^{set} respectively through the iteration procedure such as, for example,

$$j_\psi^n = -\lambda^n R\left[\hat{\beta}^n + (1-\hat{\beta}^n)\left(\frac{R_p^{ref}}{R}\right)^2\right] U(\psi), \tag{8}$$

$$\hat{\beta}^n = \hat{\beta}^{n-1} - (\beta_p^{set} - \beta_p^{n-1}), \tag{9}$$

$$\lambda^n = -I_p^{set} \Big/ \int R\left[\hat{\beta}^n + (1-\hat{\beta}^n)\left(\frac{R_p^{ref}}{R}\right)^2\right] U(\psi)\, dS, \tag{10}$$

where $\hat{\beta}^n$ and λ^n are iterative parameters for I_p and β_p respectively and $U(\psi)$ is a current density profile.

After the convergence of the minor iteration loop, the modification of the reference position R_p^{ref} is made at each iteration cycle of the major loop. According to the sign of ΔB_z^m, we move the reference position R_p^{ref} to

$$R_p^{ref,m+1} = R_p^{ref,m} - m \cdot \sigma^m \cdot \delta R, \tag{11}$$

where m is the number of the major iteration cycle, δR is a constant shift length with the positive value and σ^m is defined as

$$\begin{aligned} \sigma^m &= +1 \qquad \text{for} \quad \Delta B_z^m > 0, \\ &= -1 \qquad \text{for} \quad \Delta B_z^m < 0, \end{aligned}$$

Once the two reference positions with different signs of ΔB_z are obtained, the next reference position is adjusted by using the Regula–Falsi method. Finally, when ΔB_z or ΔR_p becomes sufficiently small, we stop the iteration and obtain the equilibrium which satisfies the constraints (a) to (c).

III. Results and Discussions

The numerical scheme presented here was incorporated into the free-boundary MHD equilibrium code SELENE 40 [9]. The virtual vacuum field numerically obtained as Eq. (5) was first compared with the analytic one given as Eq. (3) for plasmas with circular cross-section and low β_p. Figure 5 shows a good agreement

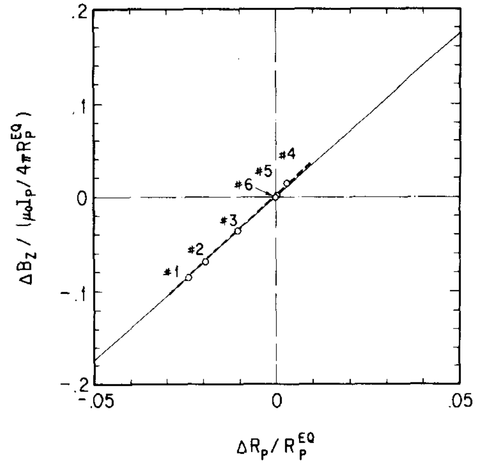

FIG. 5. Virtual vacuum field ΔB_z for displacement ΔR_p from equilibrium position R_p^{EQ}. Basic parameters are $\beta_p + l_i/2 = 0.49$, $R_p^{EQ}/a_p = 5.20$, $R_p^{EQ}/R_{out} = 0.84$ and n-index $= -(R/B_z^{ext})(\partial B_z^{ext}/\partial R) = 0$. Numerical result (dashed curve) is compared with analytic one (solid line). The circles #1 to #6 indicate sequential traces of ΔB_z with respect to the reference position set up in major loop.

between them near the equilibrium position. In the minor loop, plasma column can be regulated to given reference position with the convergence accuracy of $|(\psi^n - \psi^{n-1})/\psi_{axis}| < 10^{-6}$ after about ten interations.

The sequential traces of the virtual vacuum field for the reference position is also shown in Fig. 5. The circle #1 indicates the initial reference position and the corresponding virtual vacuum field. After the two reference positions labeled #3 with $\Delta B_z < 0$ and #4 with $\Delta B_z > 0$ are found out, the equilibrium position marked #6 can be obtained by the Regula–Falsi method.

The reference position converges to the equilibrium one with the increase in iteration of major loop as shown in Fig. 6(a). In order to get an adequate solution for plasma position, it is needed for the convergence accuracy to satisfy $|\Delta\psi^n/\psi^n_{axis}| < 1 \times 10^{-4}$ as seen from Figs. 6a and b.

The qualitative nature of this result is unchanged for (i) initial reference position far from the equilibrium one, (ii) high β_p plasmas, and (iii) external field with non-zero decay index for non-circular plasma cross section. Namely, ΔB_z increases monotonically with ΔR_p and becomes zero at $\Delta R_p = 0$ in these cases, too. As long as this dependence is held, this numerical scheme is valid for the solution procedure of plasma equilibrium with high β_p and/or non-circular cross-section. For example, the high β_p tokamak equilibrium with non-circular cross section ($\beta_p = 2.0$, aspect ratio = 4.9, ellipticity = 1.4, triangularity = 0.1 and decay-index = −2.0) was easily obtained by this method.

This numerical scheme can be also applied to the divertor configuration of which X point locates outside of the torus, as in JT-60. In this configuration, the location of the X point R_x is almost constant in the minor iteration loop and plays the same

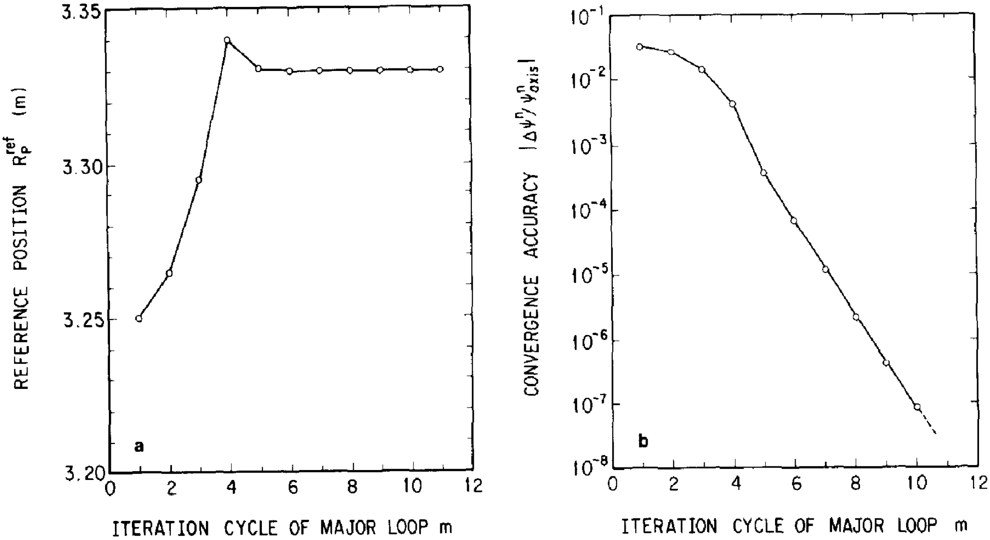

FIG. 6. Reference position set up (a) and convergence accuracy (b), as a function of iteration cycle of major loop. Basic parameters are the same as in Fig. 5. Convergence accuracy $|\Delta\psi^n/\psi^n_{\text{axis}}|$ less than 1×10^{-4} is required for an adequate solution of plasma position.

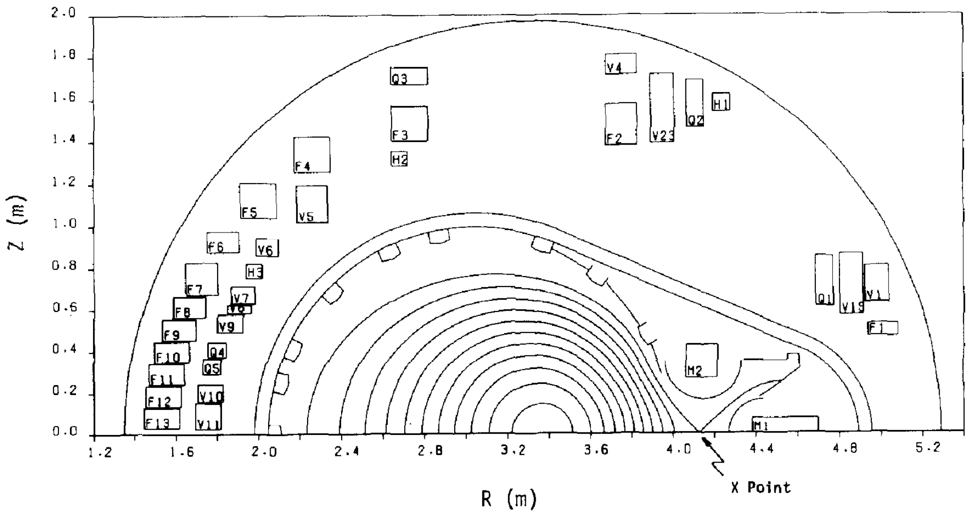

FIG. 7. Equilibrium of divertor configuration in JT-60. Basic parameters are $\beta_p + l_i/2 = 2.62$, $2I_{M1}/I_p = 0.51$, $I_{M2}/I_{M1} = -1.00$, and n-index $= 0.68$.

role as the outside limiter. This is why the plasma column is fixed at R_p^{ref} as well as the plasma current and the divertor coil current are constant. Therefore we can replace R_{out} in Eqs. (4) – (6) by R_x. The equilibria of the divertor configuration in JT-60 have been successfully obtained from low β_p plasma to high β_p plasma. Figure 7 shows a typical result of equilibrium configuration with $\beta_p = 2.1$.

It is concluded that the numerical method presented in this paper can provide a useful procedure to solve the tokamak equilibrium for a given external field in the configuration where plasma surface is always determined at an outside limiter.

In the meanwhile, for the tokamak configuration, where the plasma surface is always bounded at an inside limiter or at rail limiters parallel to the mid-plane, the equilibrium can be also solved through this numerical scheme withot the major loop. In this case, the virtual vacuum field is used as an acceleration parameter and saves the computational time by a factor of more than 3 compared with the standard method.

In this paper, the numerical scheme has been restricted to the equilibrium with an up–down symmetry. It is very easy, however, to extend our numerical scheme to the up–down asymmetric configuration such as a single null divertor. The equilibrium for a given external field will be obtained by using the virtual vacuum field with the horizontal component ΔB_R.

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